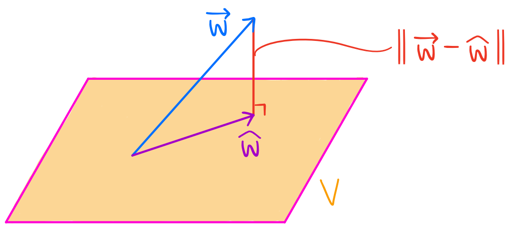
Lecture 32. Orthogonal projections

Def Consider a subspace V of $\mathbb{R}^n$ and a vector $\vec{w} \in \mathbb{R}^n$.

(1) The orthogonal projection of $\vec{w}$ onto V is the vector

$$\hat{w} = \mathrm{Proj}_V \vec{w} \in V$$

such that $\vec{w} - \hat{w}$ is orthogonal to all vectors in V.



(2) The distance from $\vec{w}$ to V is $\|\vec{w} - \hat{w}\|$.

Note (1) If $\vec{w}$ lies in V, we have $\hat{w} = \vec{w}$.

(2) $\hat{w}$ is the closest vector to $\vec{w}$ in V.

Thm If V is a subspace of $\mathbb{R}^n$ together with an orthogonal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$, for any $\vec{w} \in \mathbb{R}^n$ we have

$$\mathrm{Proj}_V \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \quad \text{with} \quad \boxed{c_i = \frac{\vec{w}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}}.$$

pf $\vec{w} - \hat{w}$ is orthogonal to all vectors in V.

$$\Longrightarrow (\vec{w} - \hat{w})\cdot\vec{v}_i = 0 \Longrightarrow \vec{w}\cdot\vec{v}_i - \hat{w}\cdot\vec{v}_i = 0 \Longrightarrow \vec{w}\cdot\vec{v}_i = \hat{w}\cdot\vec{v}_i$$

Since $\hat{w} = \mathrm{Proj}_V \vec{w}$ lies in V, we may write $\hat{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$

$$\Longrightarrow \vec{w}\cdot\vec{v}_i = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m)\cdot\vec{v}_i$$

$$\Longrightarrow \vec{w}\cdot\vec{v}_i = c_1\vec{v}_1\cdot\vec{v}_i + c_2\vec{v}_2\cdot\vec{v}_i + \cdots + c_n\vec{v}_n\cdot\vec{v}_i$$

$$\Longrightarrow \vec{w}\cdot\vec{v}_i = c_i\vec{v}_i\cdot\vec{v}_i \quad (\vec{v}_i\cdot\vec{v}_j = 0 \text{ for } i \neq j)$$

$$\Longrightarrow c_i = \frac{\vec{w}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$$

Ex Consider the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ 8 \\ -1 \\ -6 \end{bmatrix}$$

(1) Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal.

Sol $\vec{u} \cdot \vec{v} = 2 \cdot 1 + 3 \cdot 0 + 0 \cdot (-1) + 1 \cdot (-2) = 0$

$\Rightarrow$ $\vec{u}$ and $\vec{v}$ are $\boxed{\text{orthogonal}}$

(2) Find the orthogonal projection of $\vec{w}$ onto the subspace of $\mathbb{R}^4$ spanned by $\vec{u}$ and $\vec{v}$.

Sol The orthogonal projection is $\hat{w} = c_1 \vec{u} + c_2 \vec{v}$ with

$$c_1 = \frac{\vec{w} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} = \frac{5 \cdot 2 + 8 \cdot 3 + (-1) \cdot 0 + (-6) \cdot 1}{2^2 + 3^2 + 0^2 + 1^2} = \frac{28}{14} = 2,$$

$$c_2 = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{5 \cdot 1 + 8 \cdot 0 + (-1) \cdot (-1) + (-6) \cdot (-2)}{1^2 + 0^2 + (-1)^2 + (-2)^2} = \frac{18}{6} = 3.$$

$$\Rightarrow \hat{w} = 2\vec{u} + 3\vec{v} = 2 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ 6 \\ -3 \\ -4 \end{bmatrix}}$$

(3) Find the distance from $\vec{w}$ to the subspace of $\mathbb{R}^4$ spanned by $\vec{u}$ and $\vec{v}$.

Sol $\|\vec{w} - \hat{w}\| = \sqrt{(5-7)^2 + (8-6)^2 + (-1-(-3))^2 + (-6-(-4))^2} = \sqrt{16} = \boxed{4}$

<u>Ex</u> Find the distance from the point $(8, 7, 2)$ to the line L spanned by

$$\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$$

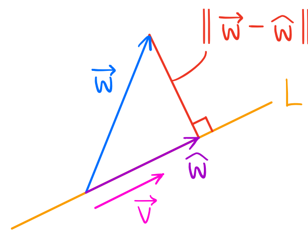
<u>Sol</u> We find the distance from $\vec{w} = \begin{bmatrix} 8 \\ 7 \\ 2 \end{bmatrix}$ to L.

The vector $\vec{v}$ gives an orthogonal basis of L.

(A set of one vector is automatically orthogonal)

The orthogonal projection of $\vec{w}$ onto L is

$$\widehat{w} = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{8\cdot 3 + 7\cdot 2 + 2\cdot(-2)}{3^2+2^2+(-2)^2}\vec{v} = 2\vec{v} = 2\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -4 \end{bmatrix}$$



$$\Longrightarrow \|\vec{w} - \widehat{w}\| = \sqrt{(8-6)^2 + (7-4)^2 + (2-(-4))^2} = \sqrt{49} = \boxed{7}$$

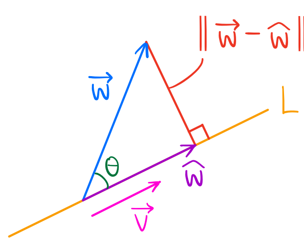
<u>Note</u> Alternatively, we may use the cross product to find

$$\|\vec{w} - \widehat{w}\| = \frac{\|\vec{v}\times\vec{w}\|}{\|\vec{v}\|}.$$

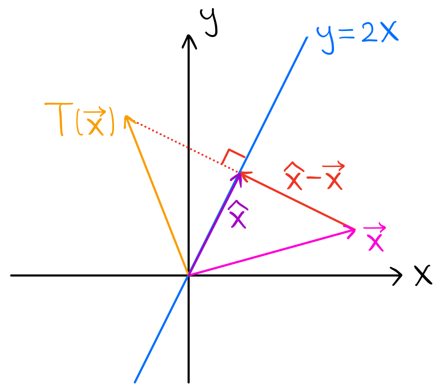
In fact, if θ is the angle between $\vec{v}$ and $\vec{w}$, we have

$$\|\vec{w} - \widehat{w}\| = \|\vec{w}\|\sin\theta = \frac{\|\vec{v}\|\,\|\vec{w}\|\sin\theta}{\|\vec{v}\|} = \frac{\|\vec{v}\times\vec{w}\|}{\|\vec{v}\|}.$$

However, the cross product is defined only for $\mathbb{R}^3$.



Ex Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which reflects each vector through the line $y = 2x$.



Sol The line $y=2x$ is spanned by $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\Longrightarrow$ $\vec{v}$ gives an orthogonal basis of the line $y=2x$.

The orthogonal projection of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ onto the line $y=2x$ is

$$\hat{x} = \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{x_1 \cdot 1 + x_2 \cdot 2}{1^2 + 2^2}\vec{v} = \frac{x_1 + 2x_2}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

Moreover, we have $T(\vec{x}) - \vec{x} = 2(\hat{x} - \vec{x})$

$$\Longrightarrow T(\vec{x}) = 2(\hat{x} - \vec{x}) + \vec{x} = 2\hat{x} - \vec{x} = \frac{2}{5}\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x_1 + 4x_2 \\ 4x_1 + 3x_2 \end{bmatrix}$$

Hence the standard matrix is

$$A = \boxed{\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}}$$

Note We have previously discussed this example in Lecture 21 using change of basis. We will revisit this example again in Lecture 34 from a slightly different perspective.